

# A SHARP REFINEMENT OF A RESULT OF ALON, BEN-SHIMON AND KRIVELEVICH ON BIPARTITE GRAPH VERTEX SEQUENCES

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**ABSTRACT.** We give a sharp refinement of a result of Alon, Ben-Shimon and Krivelevich. This gives a sufficient condition for a finite sequence of positive integers to be the vertex degree list of both parts of a bipartite graph. The condition depends only on the length of the sequence and its largest and smallest elements.

## 1. INTRODUCTION

Recall that a finite sequence  $\underline{d} = (d_1, \dots, d_n)$  of positive integers is *graphic* if there is a simple graph with  $n$  vertices having  $\underline{d}$  as its list of vertex degrees. A pair  $(\underline{d}_1, \underline{d}_2)$  of sequences (possibly of different length) is *bipartite graphic* if there is a simple, bipartite graph whose parts have  $\underline{d}_1, \underline{d}_2$  as their respective lists of vertex degrees. We say that a sequence  $\underline{d}$  is *bipartite graphic* if the pair  $(\underline{d}, \underline{d})$  is bipartite graphic; that is, if there is a simple, bipartite graph whose two parts each have  $\underline{d}$  as their list of vertex degrees. The classic Erdős–Gallai Theorem gives a necessary and sufficient condition for a sequence to be graphic. Similarly, the Gale–Ryser Theorem [5, 6] gives a necessary and sufficient condition for a pair of sequences to be bipartite graphic. In particular, the Gale–Ryser Theorem gives a necessary and sufficient condition for a single sequence to be bipartite graphic.

In [7, Theorem 6], Zverovich and Zverovich gave a sufficient condition, for a sequence to be graphic, depending only on the length of the sequence and its largest and smallest elements. A sharp refinement of this result is given in [4]. In [1, Corollary 2.2], Alon, Ben-Shimon and Krivelevich gave a result for bipartite graphic sequences, which is directly analogous to the theorem of Zverovich–Zverovich. The purpose of the present paper is to give a sharp refinement of the Alon–Ben-Shimon–Krivelevich result.

Here is the Alon–Ben-Shimon–Krivelevich result:

**Theorem 1** ([1, Corollary 2.2]). *Suppose that  $\underline{d}$  is a finite sequence of positive integers having length  $n$ , maximum element  $a$  and minimum element  $b$ . If for a real number  $x \geq 1$ , we have*

$$(1) \quad a \leq \min \left\{ xb, \frac{4xn}{(x+1)^2} \right\},$$

*then  $\underline{d}$  is bipartite graphic.*

As we will explain at the end of this introduction, Theorem 1 can be rephrased in the following equivalent form:

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**Theorem 2.** *Suppose that  $\underline{d}$  is a finite sequence of positive integers having length  $n$ , maximum element  $a$  and minimum element  $b$ . Then  $\underline{d}$  is bipartite graphic if*

$$(2) \quad nb \geq \frac{(a+b)^2}{4}.$$

The main aim of this paper is to prove the following result.

**Theorem 3.** *Suppose that  $\underline{d}$  is a finite sequence of positive integers having length  $n$ , maximum element  $a$  and minimum element  $b$ . Then  $\underline{d}$  is bipartite graphic if*

$$(3) \quad nb \geq \begin{cases} \frac{(a+b)^2}{4} & : \text{ if } a \equiv b \pmod{2}, \\ \left\lfloor \frac{(a+b)^2}{4} \right\rfloor & : \text{ otherwise,} \end{cases}$$

where  $\lfloor \cdot \rfloor$  denotes the integer part. Moreover, for any triple  $(a, b, n)$  of positive integers with  $b < a \leq n$  that fails (2), there is a non-bipartite-graphic sequence of length  $n$  with maximal element  $a$  and minimal element  $b$ .

Let us contrast the above result with the sharp result for graphic sequences given in [4]. We will require this result later in Section 5.

**Theorem 4** ([4]). *Suppose that  $\underline{d}$  is a finite sequence of positive integers having length  $n$ , maximum element  $a$  and minimum element  $b$ . Then  $\underline{d}$  is graphic if*

$$(4) \quad nb \geq \begin{cases} \left\lfloor \frac{(a+b+1)^2}{4} \right\rfloor - 1 & : \text{ if } b \text{ is odd, or } a+b \equiv 1 \pmod{4}, \\ \left\lfloor \frac{(a+b+1)^2}{4} \right\rfloor & : \text{ otherwise.} \end{cases}$$

Moreover, for any triple  $(a, b, n)$  of positive integers with  $b < a < n$  that fails (4), there is a non-graphic sequence of length  $n$  having even sum with maximal element  $a$  and minimal element  $b$ .

We give two proofs of Theorem 3. The first proof is in the spirit of the original paper of Zverovich and Zverovich, and uses the notion of *strong indices*. The preparatory results for this proof, notably Theorem 7 and Lemma 2, may be of independent interest. Our second proof is much shorter, and uses the sharp version of Zverovich–Zverovich from [4] and recent results relating bipartite graphic sequences to the degree sequences of graphs having at most one loop at each vertex [3].

The paper is organised as follows. Section 2 gives a necessary and sufficient condition for a sequence of the form  $(a^s, b^{n-s})$  to be bipartite graphic. Here and throughout the paper, the superscripts indicate the number of repetitions of the element. So, for example, the sequence  $(5, 5, 5, 4, 4)$  is denoted  $(5^3, 4^2)$ . In Section 2 we also prove Theorem 3 for sequences of the form  $(a^s, b^{n-s})$ , and we give examples showing that Theorem 3 is sharp. Section 3 presents results about bipartite graphic sequences, which are used in the first proof of Theorem 3 found in Section 4. Section 5 presents the second proof of Theorem 3.

To complete this introduction, let us establish the equivalence of Theorems 1 and 2. If  $nb \geq \frac{(a+b)^2}{4}$ , then setting  $x = \frac{a}{b}$ , we have that (1) holds. Thus Theorem 2 follows from Theorem 1. Conversely, fix  $a, b, n$  and note that the hypothesis of Theorem 1 is that  $a \leq xb$  and  $a \leq \frac{4xn}{(x+1)^2}$ . Observe that  $\frac{4xn}{(x+1)^2}$  is a monotonic decreasing function of  $x$  for  $x \geq 1$ . So if  $a \leq \frac{4xn}{(x+1)^2}$  holds for some  $x \geq \frac{a}{b}$ , then  $a \leq \frac{4xn}{(x+1)^2}$  holds for  $x = \frac{a}{b}$ , in which case (2) holds. Hence Theorem 1 follows from Theorem 2.

## 2. TWO-ELEMENT SEQUENCES

We consider two-element sequences; that is, sequences of the form  $(a^s, b^{n-s})$ .

**Theorem 5.** *Let  $a, b, n, s \in \mathbb{N}$  with  $b < a \leq n$  and  $s \leq n$ . Then the sequence  $(a^s, b^{n-s})$  is bipartite graphic if and only if  $s^2 - (a+b)s + nb \geq 0$ .*

*Proof.* We will employ [7, Theorem 8], from which we have in particular: a two-element sequence  $\underline{d} = (a^s, b^{n-s})$  is bipartite graphic if and only if

$$(5) \quad \sum_{i=1}^s (a + in_{s-i}) \leq sn \quad \text{and} \quad \sum_{i=1}^s (a + in_{n-i}) + \sum_{i=s+1}^n (b + in_{n-i}) \leq n^2,$$

where  $n_j$  is the number of elements of  $\underline{d}$  equal to  $j$ ; that is,

$$n_j = \begin{cases} s & : \text{ if } j = a \\ n - s & : \text{ if } j = b \\ 0 & : \text{ otherwise.} \end{cases}$$

Notice that the second inequality in (5) is always satisfied. Indeed,

$$\begin{aligned} \sum_{i=1}^s (a + in_{n-i}) + \sum_{i=s+1}^n (b + in_{n-i}) &= as + (n-s)b + \sum_{j=0}^{n-1} (n-j)n_j \\ &= s(a-b) + nb + (n-a)s + (n-b)(n-s) = n^2. \end{aligned}$$

So, rewriting the first inequality in (5), we have that  $\underline{d} = (a^s, b^{n-s})$  is bipartite graphic if and only if

$$(6) \quad \sum_{j=0}^{s-1} (s-j)n_j \leq s(n-a).$$

If  $b < s \leq a$ , then  $\sum_{j=0}^{s-1} (s-j)n_j = (s-b)(n-s)$  and hence

$$\sum_{j=0}^{s-1} (s-j)n_j \leq s(n-a) \iff s^2 - (a+b)s + nb \geq 0,$$

as required. It remains to consider the cases  $s \leq b$  and  $a < s$ . If  $s \leq b$ , then

$$\sum_{j=0}^{s-1} (s-j)n_j = 0 \leq s(n-a).$$

If  $a < s$ , then

$$\sum_{j=0}^{s-1} (s-j)n_j = (s-a)s + (s-b)(n-s) = s(n-a) - b(n-s) \leq s(n-a).$$

The inequality  $s^2 - (a+b)s + nb \geq 0$  holds in both these cases. Indeed, the minimum of the function  $f(s) = s^2 - (a+b)s + nb$  occurs at  $s = \frac{a+b}{2}$  so  $f(s)$  is decreasing for  $s \leq b$ , and increasing for  $a < s$ , and  $f(a) = f(b) = (n-a)b \geq 0$ .  $\square$

**Example 1.** First assume  $a \equiv b \pmod{2}$  and  $4nb < (a+b)^2$ . Then the sequence

$$(a^{\frac{a+b}{2}}, b^{\frac{2n-a-b}{2}})$$

is not bipartite graphic by Theorem 5. Now assume  $a \not\equiv b \pmod{2}$  and  $4nb < (a+b)^2 - 1$ . Then

$$(a^{\frac{a+b+1}{2}}, b^{\frac{2n-a-b-1}{2}})$$

is not bipartite graphic, again by Theorem 5. These examples show that the bound given in Theorem 3 is sharp.

**Remark 1.** Note that for two-element sequences, we can deduce Theorem 3 from Theorem 5. Indeed, suppose that  $\underline{d} = (a^s, b^{n-s})$  and that

$$nb \geq \begin{cases} \frac{(a+b)^2}{4} & : \text{ if } a \equiv b \pmod{2}, \\ \left\lfloor \frac{(a+b)^2}{4} \right\rfloor & : \text{ otherwise.} \end{cases}$$

As we observed in the proof of Theorem 5, the minimum of the function  $f(s) = s^2 - (a+b)s + nb$  occurs at  $\frac{a+b}{2}$ . If  $a+b$  is even, then

$$f(s) \geq f\left(\frac{a+b}{2}\right) = nb - \frac{(a+b)^2}{4} \geq 0,$$

and so  $\underline{d}$  is bipartite graphic by Theorem 5. So we may suppose that  $a+b$  is odd. Then as  $s$  is an integer,

$$f(s) \geq f\left(\frac{a+b-1}{2}\right) = nb - \frac{(a+b)^2 - 1}{4} = nb - \left\lfloor \frac{(a+b)^2}{4} \right\rfloor \geq 0.$$

Hence  $\underline{d}$  is bipartite graphic by Theorem 5.

### 3. STRONG INDICES

In this section,  $\underline{d} = (d_1, \dots, d_n)$  is a decreasing sequence of positive integers and for each integer  $j$ , the number of elements in  $\underline{d}$  equal to  $j$  is denoted  $n_j$ . As a particular case of [7, Theorem 7], one has the following.

**Theorem 6** ([7]). *The sequence  $\underline{d}$  is bipartite graphic if and only if  $\sum_{i=1}^k (d_i + in_{k-i}) \leq kn$ , for all indices  $k$ .*

Recall the following standard definition.

**Definition 1.** In the sequence  $\underline{d}$ , an index is said to be *strong* if  $d_k \geq k$ .

The following result improves Theorem 6.

**Theorem 7.** *The sequence  $\underline{d}$  is bipartite graphic if and only if  $\sum_{i=1}^k (d_i + in_{k-i}) \leq kn$ , for all strong indices  $k$ .*

*Proof.* Necessity follows from Theorem 7 in [7]. To prove sufficiency, define

$$F_k = kn - \sum_{i=1}^k (d_i + in_{k-i}) = kn - \sum_{i=1}^k d_i - \sum_{i=0}^k (k-i)n_i.$$

Suppose that  $F_k \geq 0$  for all strong indices  $k$ . We will show that  $F_k \geq 0$  for all indices  $k$ . To do this, we show that the minimum value of  $F_k$ , for  $k = 1, 2, \dots, n$ , is nonnegative, and to do this we look at the smallest  $k$  for which  $F_k$  assumes the minimum value. Thus it suffices to show that  $F_1$  and  $F_n$  are nonnegative and  $F_k \geq 0$  for all  $k = 2, \dots, n-1$  such that  $F_{k-1} > F_k$  and  $F_{k+1} \geq F_k$ . We will make use of the following lemma. Define the function  $f : \mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \cup \{0\}$  as follows:  $f(k) = \max\{p : d_p \geq k+1\}$ , with the convention that  $\max \emptyset = 0$ .

**Lemma 1.** *For the sequence  $\underline{d}$ , suppose that  $n \geq d_1$ . For a given  $k = 0, 1, \dots, n$ , denote  $p = f(k)$ . Then, in the above notation,*

- (a) *if  $k, p > 0$ , then at least one of them is a strong index,*
- (b)  $\sum_{s=k+1}^n n_s = p$  *and*  $\sum_{s=0}^n n_s = n$ ,
- (c)  $\sum_{s=k+1}^n sn_s = \sum_{i=1}^p d_i$  *and*  $\sum_{s=0}^n sn_s = \sum_{i=1}^n d_i$ ,
- (d)  $F_k = \sum_{i=1}^n d_i - \sum_{i=1}^k d_i - \sum_{i=1}^p d_i + kp$ . *In particular, if  $f(p) = k$ , then  $F_k = F_p$ .*

*Proof.* (a) Suppose  $k$  is not a strong index, so that  $k > d_k$ . As  $p = f(k)$  is assumed to be positive we have  $p \in \{1, \dots, n\}$  and moreover,  $d_p \geq k+1 > d_k$ . So, as  $\underline{d}$  is decreasing,  $p < k$ . Thus  $d_p \geq k+1 > p$  and so  $p$  is a strong index, as required.

(b) The left-hand side of the first equality equals  $\#\{s : d_s \geq k+1\} = p$  by definition. The second equality is obvious.

(c) For an arbitrary  $s \geq 0$  we have  $sn_s = \sum_{i:d_i=s} d_i$ . It follows that  $\sum_{s=k+1}^n sn_s = \sum_{s=k+1}^n \sum_{i:d_i=s} d_i = \sum_{i:d_i \geq k+1} d_i = \sum_{i=1}^p d_i$ . This proves the first equality; the second equality is obvious.

(d) We have by (b) and (c):

$$\begin{aligned} F_k &= kn - \sum_{i=1}^k d_i - k \sum_{i=0}^k n_i + \sum_{i=0}^k in_i \\ &= k \left( n - \sum_{i=0}^k n_i \right) - \sum_{i=1}^k d_i + \sum_{i=0}^n in_i - \sum_{i=k+1}^n in_i \\ &= kp - \sum_{i=1}^k d_i + \sum_{i=1}^n d_i - \sum_{i=1}^p d_i, \end{aligned}$$

as required. If not only  $f(k) = p$ , but also  $f(p) = k$ , then  $F_k = F_p$ , as the latter expression for  $F_k$  is symmetric with respect to  $k$  and  $p$ .  $\square$

Continuing with the proof of the theorem, by Lemma 1(b),

$$(7) \quad F_{k+1} - F_k = n - d_{k+1} - \sum_{i=0}^k n_i = \sum_{i=k+1}^n n_i - d_{k+1} = f(k) - d_{k+1}.$$

Moreover,  $F_n = n^2 - \sum_{i=1}^n d_i - n \sum_{i=0}^n n_i + \sum_{i=0}^n in_i = 0$  by Lemma 1(b, c) and  $F_1 \geq 0$  by assumption, as  $d_1 \geq 1$ . By (7) and Lemma 1(b), the inequalities  $F_{k-1} > F_k$  and  $F_{k+1} \geq F_k$  give

$$\begin{aligned} F_{k+1} - F_k &= f(k) - d_{k+1} \geq 0, \\ F_k - F_{k-1} &= f(k-1) - d_k = f(k) + n_k - d_k < 0. \end{aligned}$$

That is,

$$(8) \quad d_{k+1} \leq f(k) < d_k - n_k.$$

Let  $k$  be a non-strong index for which (8) holds. Denote  $p = f(k)$ . If  $p > 0$ , then  $p$  is a strong index by Lemma 1(a), hence  $F_p \geq 0$  by assumption. Moreover, by (8) we have  $d_{k+1} \leq p$  and  $d_k > p + n_k$  so  $d_k \geq p + 1$  and  $d_{k+1} < p + 1$ . It follows that  $k = \max\{s : d_s \geq p + 1\}$ , so  $f(p) = k$  by definition. Then, by Lemma 1(d), we have  $F_k = F_p \geq 0$ . So we may assume that  $p = 0$ . Then  $d_{k+1} = 0$ , by (8), and hence  $d_j = 0$  for all  $j > k$ . Furthermore, as  $f(k) = p = 0$ , we have  $\{s : d_s \geq k + 1\} = \emptyset$ , and so  $n_i = 0$  for all  $i > k$ . So by (7), for every  $j > k$  we have  $F_j - F_{j-1} = \sum_{i=j}^n n_i - d_j = 0$ . Thus  $F_k = F_n$ . As  $F_n = 0$  from the above, we get  $F_k = 0$ , as required.  $\square$

In the next section, we will also need the following lemma, which is a variation of [4, Lemma 1].

**Lemma 2.** *Suppose that  $\underline{d}$  has maximum element  $a = d_1 \leq n$  and minimum element  $b = d_n$ . For every strong index  $k > b$ , we have*

$$\sum_{i=1}^k (d_i + in_{k-i}) \leq n(k - b) + K(a + b) - K^2,$$

where  $K$  is the largest strong index,  $K = \max\{k : d_k \geq k\}$ .

*Proof.* Let  $k > b$  be a strong index. We have  $\sum_{i=1}^k d_i \leq ka$ . Furthermore, since  $n_j = 0$  for  $j < b$ , we have

$$\sum_{i=1}^k in_{k-i} = \sum_{j=0}^{k-1} (k-j)n_j \leq (k-b) \sum_{j=0}^{k-1} n_j.$$

The sum  $\sum_{j=0}^{k-1} n_j$  counts the number of elements of  $\underline{d}$  strictly less than  $k$ , hence  $\sum_{j=0}^{k-1} n_j \leq n - K$  as  $d_K \geq K \geq k$ . Hence

$$(9) \quad \sum_{i=1}^k (d_i + in_{k-i}) \leq ka + (k-b)(n-K).$$

As  $a \geq d_K \geq K$ , we have  $a + 1 - K \geq 1$ . Thus, using  $k \leq K$ , inequality (9) gives

$$\begin{aligned} \sum_{i=1}^k (d_i + in_{k-i}) &\leq ka + (k-b)(n-K) = kn + k(a-K) + bK - bn \\ &\leq kn + K(a-K) + bK - bn \\ &= n(k-b) + K(a+b) - K^2, \end{aligned}$$

as required.  $\square$

#### 4. FIRST PROOF OF THEOREM 3

Let  $\underline{d}$  be a sequence satisfying hypothesis (3) of Theorem 3. If  $a \equiv b \pmod{2}$ , then the result follows from Theorem 2. So we may assume that  $a, b$  have different parity. Let  $k$  be a strong index and suppose first that  $k > b$ . By Lemma 2,

$$(10) \quad \sum_{i=1}^k (d_i + in_{k-i}) \leq n(k-b) + K(a+b) - K^2,$$

where  $K$  denotes the largest strong index. As a quadratic in  $K$ , the maximal value of  $n(k-b) + K(a+b) - K^2$  is attained at  $K = \frac{a+b \pm 1}{2}$  and

$$n(k-b) + \frac{(a+b \pm 1)}{2}(a+b) - \left(\frac{a+b \pm 1}{2}\right)^2 = n(k-b) + \frac{1}{4}(a+b)^2 - \frac{1}{4}.$$

Hence, since  $nb \geq \left\lfloor \frac{(a+b)^2}{4} \right\rfloor = \frac{(a+b)^2}{4} - \frac{1}{4}$ , we have

$$n(k-b) + K(a+b) - K^2 \leq n(k-b) + \frac{1}{4}(a+b)^2 - \frac{1}{4} \leq kn.$$

So by (10), we have  $\sum_{i=1}^k (d_i + in_{k-i}) \leq kn$  and hence  $\underline{d}$  is bipartite graphic by Theorem 7. On the other hand, if  $k \leq b$ , then  $\underline{d}$  contains no elements less than  $k$  and hence

$$(11) \quad \sum_{i=1}^k (d_i + in_{k-i}) = \sum_{i=1}^k d_i \leq ka.$$

Note that  $n \geq a$ , since otherwise by (3), we would have  $ab > nb \geq \frac{(a+b)^2-1}{4}$ , and hence  $(a-b)^2 < 1$ , giving  $a = b$ , which is impossible as  $a, b$  have different parity. So (11) gives  $\sum_{i=1}^k (d_i + in_{k-i}) \leq kn$  and once again,  $\underline{d}$  is bipartite graphic by Theorem 7.

#### 5. SECOND PROOF OF THEOREM 3

Suppose we have a decreasing sequence  $\underline{d} = (a, \dots, b)$  of length  $n$ , and suppose it satisfies hypothesis (3) of Theorem 3. By Remark 1, we may assume that  $\underline{d}$  has at least 3 distinct elements. Suppose that  $n_a = s$ ; that is,  $\underline{d}$  has precisely  $s$  elements equal to  $a$ . Now consider the sequence  $\underline{d}'$  obtained from  $\underline{d}$  by reducing the first  $s$  elements of  $\underline{d}$  by 1. So  $\underline{d}'$  has maximal element  $a' = a - 1$ . Note that  $\underline{d}$  has at least 3 distinct elements, hence the minimum element of  $\underline{d}'$  is still  $b$ . Suppose for the moment that  $\underline{d}'$  has even sum. We will show that  $\underline{d}'$  is graphic. From (3), we have

$$nb \geq \begin{cases} \frac{(a+b)^2}{4} & : \text{ if } a \equiv b \pmod{2}, \\ \left\lfloor \frac{(a+b)^2}{4} \right\rfloor & : \text{ otherwise,} \end{cases}$$

We will show that

$$(12) \quad nb \geq \begin{cases} \left\lfloor \frac{(a' + b + 1)^2}{4} \right\rfloor - 1 & : \text{ if } b \text{ is odd, or } a' + b \equiv 1 \pmod{4}, \\ \left\lfloor \frac{(a' + b + 1)^2}{4} \right\rfloor & : \text{ otherwise,} \end{cases}$$

from which we can conclude that  $\underline{d}'$  is graphic by Theorem 4. Consider two cases according to whether or not  $a \equiv b \pmod{2}$ . If  $a \equiv b \pmod{2}$ , then our hypothesis is  $nb \geq \frac{(a+b)^2}{4}$ , and hence

$$nb \geq \frac{(a' + b + 1)^2}{4} = \left\lfloor \frac{(a' + b + 1)^2}{4} \right\rfloor,$$

and so (12) holds. Similarly, if  $a \not\equiv b \pmod{2}$ , then our hypothesis is  $nb \geq \left\lfloor \frac{(a+b)^2}{4} \right\rfloor$ , and hence

$$nb \geq \left\lfloor \frac{(a' + b + 1)^2}{4} \right\rfloor,$$

and again (12) holds. Thus in either case,  $\underline{d}'$  is graphic.

We now use a result of [3]. By a *graph-with-loops* we mean a graph, without multiple edges, in which there is at most one loop at each vertex. For a graph-with-loops, the *reduced degree* of a vertex is taken to be the number of edges incident to the vertex, with *loops counted once*. This differs from the usual definition of degree in which each loop contributes two to the degree. By [3, Corollary 1], a sequence  $\underline{d}$  of positive integers is the sequence of reduced degrees of the vertices of a graph-with-loops if and only if  $\underline{d}$  is bipartite graphic. In our case,  $\underline{d}'$  is graphic. Take a realization of  $\underline{d}'$  as the degree sequence of some graph  $G'$ , and label the vertices of  $G'$  in the same order as  $\underline{d}'$ . Now add a loop to each of the first  $s$  nodes of  $G'$  and call the resulting graph-with-loops  $G$ . So the sequence of reduced degrees of  $G$  is  $\underline{d}$ . Thus by [3, Corollary 1],  $\underline{d}$  is bipartite graphic.

It remains to deal with the case where  $\underline{d}'$  has odd sum. Since  $\underline{d}$  has at least 3 distinct elements, we can modify the above construction as follows: we take the sequence  $\underline{d}''$  obtained from  $\underline{d}$  by reducing the first  $(s + 1)$  elements of  $\underline{d}$  by 1. Then  $\underline{d}''$  has even sum, maximum element  $a - 1$  and minimum element  $b$ , and we proceed exactly as above, only adding  $s + 1$  loops.

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